

Enveloping actions for twisted partial actions

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Abstract Let $A\#_{\alpha,\omega}H$ be a partial crossed product. In this paper, we first generalize the theorem about the existence of an enveloping action to twisted partial actions. Second, we construct a Morita context between the partial crossed product and the crossed product related to the enveloping action. Furthermore, we discuss equivalences of partial crossed products. Finally, we investigate when $A \subset A\#_{\alpha,\omega}H$ becomes a separable extension.

Key words Enveloping action, partial crossed product, Morita context, separable extension.

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1 Introduction

Partial actions of groups have been introduced in the theory of operator algebras as a general approach to study C^* -algebras by partial isometries in [16]. A treatment from a purely algebraic point of view was given recently in [9]-[13]. In particular, the Galois theory of partial group actions developed in [9] inspired further Galois theoretic results in [6], as well as the introduction and study of partial Hopf actions and coactions in [7]. The latter paper became in turn the starting point for further investigation of partial Hopf (co)actions in [1]-[4]. The general notion of a (continuous) twisted partial action of a locally compact group on a C^* -algebra (a twisted partial C^* -dynamical system) and the corresponding crossed products were given by Exel in [17]. Twisted partial actions of groups on abstract rings and corresponding crossed products were recently introduced in [12]. More recent algebraic results on twisted partial actions and corresponding crossed products were obtained in [13] and [5]. The algebraic concept of twisted partial actions also motivated the study of projective partial group representations, the corresponding

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partial Schur Multiplier and the relation to partial group actions with K -valued twistings in [14] and [15], contributing towards the elaboration of a background for a general cohomological theory based on partial actions. Further information around partial actions may be consulted in the survey [8]. As a unified approach for twisted partial group actions, partial Hopf actions and twisted actions of Hopf algebras, the notion of a twisted partial Hopf action is introduced in [4].

Certainly, the theory of partial actions of Hopf algebras remains as a huge landscape to be explored. Alves and Batista [2] generalized the theorem about the existence of an enveloping action, also known as the globalization theorem and they constructed a Morita context between the partial smash product and the smash product related to the enveloping action, and this present work intends to generalize some results for partial group actions and partial Hopf actions to the context of twisted partial actions.

This paper is organized as follows:

In Section 3, we prove the existence of an enveloping action for such a twisted partial actions. In Section 4, we construct a Morita context between the partial crossed product $A \#_{\alpha, \omega} H$ and the crossed product $B \#_u H$, where H is a Hopf algebra which acts partially on the unital algebra A , B is an enveloping action for twisted partial actions. This result can also be found in [2] for the context of partial group actions. In Section 5, we shall discuss equivalences of partial crossed products and this result is similar to [4]. In Section 6, we investigate when $A \subset A \#_{\alpha, \omega} H$ becomes a separable extension.

2 Preliminaries

Throughout the paper, let k be a fixed field and all algebraic systems are supposed to be over k . Let M be a vector space over k and let id_M the usual identity map. For the comultiplication Δ in a coalgebra C with a counit ε_C , we use the Sweedler-Heyneman's notation (see Sweedler [20]): $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$.

We first recall some basic results and propositions that we will need later from Alves and Batista [1],[2].

2.1. Partial module algebra Let H be a Hopf algebra and A an algebra. A is said to be a partial H -module algebra if there exists a k -linear map $\cdot \{ \cdot : H \otimes A \rightarrow A \}$ satisfying the following conditions:

$$\begin{aligned} h \cdot (ab) &= (h_{(1)} \cdot a)(h_{(2)} \cdot b), \\ 1_H \cdot a &= a, \\ h \cdot (g \cdot a) &= (h_{(1)} \cdot 1_A)(h_{(2)} g \cdot a), \end{aligned}$$

for all $h, g \in H$ and $a, b \in A$.

3 Enveloping actions

Recall from [4] that the definition of the partial crossed product. A twisted partial action of H on A is a pair (α, ω) , where $\alpha : H \otimes A \rightarrow A, \alpha(h \otimes a) = h \cdot a$ and $\omega : H \otimes H \rightarrow A, \omega(h \otimes g) = \omega(h, g)$ be two linear maps such that the following conditions hold:

$$1_H \cdot a = a, \quad (3.1)$$

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b), \quad (3.2)$$

$$(h_{(1)} \cdot (l_{(1)} \cdot a))\omega(h_{(2)}, l_{(2)}) = \omega(h_{(1)}, l_{(1)})(h_{(2)}l_{(2)} \cdot a), \quad (3.3)$$

$$\omega(h, l) = \omega(h_{(1)}, l_{(1)})(h_{(2)}l_{(2)} \cdot 1_A), \quad (3.4)$$

for all $a, b \in A$ and $h, l \in A$.

Lemma 3.1.^[4] If (α, ω) is a twisted partial action, then the following identities hold:

$$\omega(h, l) = (h_{(1)} \cdot (l_{(1)} \cdot 1_A))\omega(h_{(2)}, l_{(2)}) = (h_{(1)} \cdot 1_A)\omega(h_{(2)}, l), \quad (3.5)$$

for all $h, l \in H$.

We say that the map ω is trivial, if the following condition holds:

$$h \cdot (l \cdot 1_A) = \omega(h, l) = (h_{(1)} \cdot 1_A)(h_{(2)}l \cdot 1_A),$$

for all $h, l \in H$. In this case, the twisted partial action (α, ω) turns out to be a partial action of H on A .

Given any twisted partial action (α, ω) of H on A , we can define on the vector space $A \otimes H$ a product, given by the multiplication

$$(a \otimes h)(b \otimes l) = a(h_{(1)} \cdot b)\omega(h_{(2)}, l_{(1)}) \otimes h_{(3)}l_{(2)},$$

for all $a, b \in A$ and $h, l \in H$. Denote $A \#_{\alpha, \omega} H$ to be the subspace of $A \otimes H$ generated by the elements of the form $a \# h = a(h_{(1)} \cdot 1_A) \otimes h_{(2)}$, for all $a \in A$ and $h \in H$. Recall from [] that $A \#_{\alpha, \omega} H$ is associative via the multiplication Eq.(3.7) and $A \#_{\alpha, \omega} H$ is unital with $1_A \# 1_H = 1_A \otimes 1_H$ if and only if

$$\omega(1_H, h) = \omega(h, 1_H) = h \cdot 1_A, \quad (3.6)$$

$$(h_{(1)} \cdot (l_{(1)} \cdot a))\omega(h_{(2)}, l_{(2)}) = \omega(h_{(1)}, l_{(1)})(h_{(2)}l_{(2)} \cdot a), \quad (3.7)$$

$$(h_{(1)} \cdot \omega(l_{(1)}, m_{(1)}))\omega(h_{(2)}, l_{(2)}m_{(2)}) = \omega(h_{(1)}, l_{(1)})\omega(h_{(2)}l_{(2)}, m_{(2)}), \quad (3.8)$$

for all $h, l, m \in H$ and $a \in A$.

Recall from [4] that let B be a unital k -algebra measured by an action $\beta : H \otimes B \rightarrow B$,

denoted by $\beta(h \otimes b) = h \triangleright b$, which is twisted by a map $u : H \otimes H \rightarrow B$, i.e.,

$$\begin{aligned} 1_H \triangleright a &= a, \\ h \triangleright (ab) &= (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \\ h \triangleright 1_A &= \varepsilon(h)1_A, \\ u(1_H, h) &= u(h, 1_H) = \varepsilon(h)1_A, \\ (h_{(1)} \triangleright (l_{(1)} \triangleright a))u(h_{(2)}, l_{(2)}) &= u(h_{(1)}, l_{(1)})(h_{(2)}l_{(2)} \triangleright a), \\ u(h_{(1)}, l_{(1)})u(h_{(2)}l_{(2)}, m) &= (h_{(1)} \triangleright u(l_{(1)}, m_{(1)}))u(h_{(2)}, l_{(2)}m_{(2)}), \end{aligned}$$

for all $h, l, m \in H$ and $a \in A$. Suppose that 1_A is a non-trivial central idempotent of B , and let A be the ideal generated by 1_A . Given $a \in A$ and $h \in H$, define a map $\cdot : H \otimes A \rightarrow A$ by $h \cdot a = 1_A(h \triangleright a)$. We say that the partial action $h \cdot a = 1_A(h \triangleright a)$ and $\omega(h, l) = (h_{(1)} \cdot 1_A)u(h_{(2)}, l_{(1)})(h_{(3)}l_{(2)} \cdot 1_A)$ is the twisted partial action induced by B .

Definition 3.2.^[4] With notations as above. A morphism of algebras $\theta : A \rightarrow B$ is said to be a morphism of partial H -module algebras if $\theta(h \cdot a) = h \cdot \theta(a)$ for all $h, k \in H$ and $a \in A$. If, in addition, θ is an isomorphism, the partial actions are called equivalent.

Recall from [2] that if B is an H -module algebra and A is a right ideal of B with unity 1_A , the induced partial action on A is called admissible if $B = H \triangleright A$.

Definition 3.3. With notations as above. An enveloping action for A is a pair (B, θ) , where

- (a) B is a unital k -algebra measured by an action $\beta : H \otimes B \rightarrow B$;
- (b) The map $\theta : A \rightarrow B$ is a monomorphism of algebras;
- (c) The sub-algebra $\theta(A)$ is an ideal in B ;
- (d) The partial action on A is equivalent to the induced partial action on $\theta(A)$;
- (e) The induced partial action on $\theta(A)$ is admissible.
- (d) $\theta(a\omega(g, h)) = \theta(a)u(g, h)$, $\theta(\omega(g, h)a) = u(g, h)\theta(a)$, for any $g, h \in H$ and $a \in A$.

4 A Morita context

In this section, we will construct a Morita context between the partial crossed product $A \#_{\alpha, \omega} H$ and the crossed product $B \#_u H$, where B is an enveloping action for the partial crossed product.

Lemma 4.1. *Let (α, ω) be a twisted partial action and (B, θ) an enveloping action, then there is an algebra monomorphism from the partial crossed product $A \#_{\alpha, \omega} H$ into the crossed product $B \#_u H$.*

Proof. Define $\Phi : A \#_{\alpha, \omega} H \rightarrow B \#_u H$ by $a \otimes h \mapsto \theta(a) \otimes h$ for $h, g \in H$ and $a, b \in A$.

We first check that Φ is a morphism of algebras as follows:

$$\begin{aligned}
\Phi((a \otimes h)(b \otimes g)) &= \Phi(a(h_{(1)} \cdot b)\omega(h_{(2)}, g_{(1)}) \otimes h_{(3)}g_{(2)}). \\
&= \theta(a(h_{(1)} \cdot b)\omega(h_{(2)}, g_{(1)}))\#h_{(3)}g_{(2)} \\
&= \theta(a)(h_{(1)} \cdot \theta(b))u(h_{(2)}, g_{(1)})\#h_{(3)}g_{(2)} \\
&= (\theta(a)\#h)(\theta(b)\#g) \\
&= \Phi(a \otimes h)\Phi(b \otimes g).
\end{aligned}$$

Next, we will verify that Φ is injective. For this purpose, take $x = \sum_{i=1}^n a_i \otimes h_i \in \ker \Phi$ and choose $\{a_i\}_{i=1}^n$ to be linearly independent. Since θ is injective, we conclude that $\theta(a_i)$ are linearly independent. For each $f \in H^*$, $\sum_{i=1}^n \theta(a_i)f(h_i) = 0$, it follows that $f(h_i) = 0$, so $h_i = 0$. Therefore we have $x = 0$ and Φ is injective, as desired.

Since the partial crossed product $A\#_{\alpha,\omega}H$ is a subalgebra of $A \otimes H$, it is injectively mapped into $B\#_uH$ by Φ . A typical element of the image of the partial crossed product is

$$\begin{aligned}
\Phi((a \otimes h)(1_A \otimes 1_H)) &= \Phi(a \otimes h)\Phi(1_A \otimes 1_H) \\
&= (\theta(a)\#h)(\theta(1_A)\#1_H) \\
&= \theta(a(h_{(1)} \cdot \theta(1_A)))\#h_{(2)}.
\end{aligned}$$

And this completes the proof. \square

Take $M = \Phi(A \otimes H) = \{\sum_{i=1}^n \theta(a_i)\#h_i; a_i \in A\}$ and take N as the subspace of $B\#_uH$ generated by the elements $h_{(1)} \cdot \theta(a) \otimes h_{(2)}$ with $h \in H$ and $a \in A$.

Proposition 4.2. Let (α, ω) be a twisted partial action and (B, θ) an enveloping action. Suppose that $\theta(A)$ is an ideal of B , then M is a right $B\#_uH$ module and N is a left $B\#_uH$ module.

Proof. In order to prove M is a right $B\#_uH$ module, let $\theta(a)\#h \in M$ and $b\#k \in B\#_uH$, then

$$(\theta(a)\#h)(b\#k) = \theta(a)(h_{(1)} \triangleright b)u(h_{(2)}, k_{(1)})\#h_{(3)}k_{(2)}.$$

Which lies in $\Phi(A \otimes H)$ because $\theta(A)$ is an ideal in B .

Now we show that N is a left $B\#H$ module is similar to [2]. \square

Now, the left $A\#_{\alpha,\omega}H$ module structure on M and a right $A\#_{\alpha,\omega}H$ module structure on N induced by the monomorphism Φ is in [2].

Proposition 4.3. Under the same hypotheses of Proposition 4.2, M is indeed a left $A\#_{\alpha,\omega}H$ module with the map \blacktriangleright and N is a right $A\#_{\alpha,\omega}H$ module with the map \blacktriangleleft .

Proof. We first claim that $A\#_{\alpha,\omega}H \blacktriangleright M \subseteq M$. In fact,

$$\begin{aligned}
& (a(h_{(1)} \cdot 1_A) \otimes h_{(2)}) \blacktriangleright (\theta(b))\#k \\
&= (\theta(a)(h_{(1)} \cdot \theta(1_A))\#h_{(2)})(\theta(b))\#k \\
&= \theta(a)(h_{(1)} \cdot \theta(1_A))(h_{(2)} \triangleright \theta(b))u(h_{(3)}, k_{(1)})\#h_{(4)}k_{(2)} \\
&= \theta(a)(h_{(1)} \cdot \theta(1_A))(h_{(2)} \cdot \theta(b))u(h_{(3)}, k_{(1)})\#h_{(4)}k_{(2)} \\
&= \theta(a)(h_{(1)} \cdot \theta(b))u(h_{(2)}, k_{(1)})\#h_{(3)}k_{(2)}.
\end{aligned}$$

Which lies inside M because $\theta(A)$ is an ideal of B . \square

Next, we verify that $N \blacktriangleleft A\#_{\alpha,\omega}H \subseteq N$, which is similarly to N is a left $B\#_uH$ module. which holds because $\theta(1_A)$ is a central idempotent.

The last ingredient for a Morita context is to define two bimodule morphisms

$$\sigma : N \otimes_{A\#_{\alpha,\omega}H} M \rightarrow B\#_uH \quad \text{and}$$

$$\tau : M \otimes_{B\#_uH} N \rightarrow A\#_{\alpha,\omega}H \cong \Phi(A\#_{\alpha,\omega}H).$$

As M, N and $A\#_{\alpha,\omega}H$ are viewed as subalgebras of $B\#H$, these two maps can be taken as the usual multiplication on $B\#_uH$. The associativity of the product assures us that these maps are bimodule morphisms and satisfy the associativity conditions.

Proposition 4.4. The partial crossed product $A\#_{\alpha,\omega}H$ is Morita equivalent to the crossed product $B\#_uH$.

5 Equivalences of partial crossed products

In this section, we shall discuss equivalences of partial crossed products and this result is similar to [4]. From now on, unless explicitly stated, we always assume partial actions of a Hopf algebra H over an algebra A such that the map $e \in \text{Hom}(H, A)$, given by $e(h) = h \cdot 1_A$, is central with respect to the convolution product. Given a twisted partial H -module algebra $A = (A, \alpha, \omega)$, we define two linear maps $f_1, f_2 : H \otimes H \rightarrow A$ as follows:

$$f_1(h, k) = (h \cdot 1_A)\varepsilon(k), \quad f_2(h, k) = hk \cdot 1_A.$$

Definition 5.1.^[4] Let $A = (A, \alpha, \omega)$ be a twisted partial H -module algebra. We will say that the partial action is symmetric, if

- (1) f_1, f_2 are central in $\text{Hom}(H \otimes H, A)$,
- (2) ω satisfies the conditions Eq.(3.7) and Eq.(3.8) and has a convolution inverse ω' in the ideal $\langle f_1 * f_2 \rangle \in \text{Hom}(H \otimes H, A)$,
- (3) $h \cdot (k \cdot 1_A) = (h_{(1)} \cdot 1_A)(h_{(2)}k \cdot 1_A)$ for any $h, k \in H$.

Definition 5.2. If there exists linear maps $u, v \in \text{Hom}(H, A)$ satisfy $(u * v)(h) = (v * u)(h) = h \cdot 1_A$, $u(h) = u(h_{(1)})(h_{(2)} \cdot 1_A) = (h_{(1)} \cdot 1_A)u(h_{(2)})$ and $u(1_H) = v(1_H) = 1_A$. Then we call v is a weak convolution-invertible linear map and $u = v^{-1}$.

Let H be a Hopf algebra and (A, α, ω) a symmetric twisted partial H -module algebra, and $v \in \text{Hom}(H, A)$ a weak convolution-invertible linear map. Define $\omega^v : H \otimes H \rightarrow A$ and weakly action of H on A by

$$\omega^v(h, g) = v(h_{(1)})(h_{(2)} \cdot v(g_{(1)}))\omega(h_{(3)}, g_{(2)})v^{-1}(h_{(4)}g_{(3)})$$

and

$$h \cdot^v a = v(h_{(1)})(h_{(2)} \cdot a)v^{-1}(h_{(3)})$$

for any $h, g \in H$ and $a \in A$.

Lemma 5.3. With the notations as above, let H be a Hopf algebra and (A, α, ω) a symmetric twisted partial H -module algebra. Then $\omega^{vu} = (\omega^u)^v$ and $\cdot^{vu} = (\cdot^u)^v$, where $u, v \in \text{Hom}(H, A)$ are weak convolution-invertible linear maps.

Proof. For any $h, g \in H, a \in A$, we have we have

$$\begin{aligned} & \omega^{vu}(h, g) \\ &= (vu(h_{(1)})(h_{(2)} \cdot (vu(g_{(1)})))\omega(h_{(3)}, g_{(2)})(vu)^{-1}(h_{(4)}g_{(3)}) \\ &= v(h_{(1)})u(h_{(2)})(h_{(3)} \cdot v(g_{(1)}))(h_{(4)} \cdot u(g_{(2)}))\omega(h_{(5)}, g_{(3)})u^{-1}(h_{(6)}g_{(4)})v^{-1}(h_{(7)}g_{(5)}) \\ &= v(h_{(1)})u(h_{(2)})(h_{(3,1)} \cdot v(g_{(1)}))u^{-1}(h_{(4)})u(h_{(5)})(h_{(6)} \cdot u(g_{(2)})) \\ & \quad \omega(h_{(7)}, g_{(3)})u^{-1}(h_{(8)}g_{(4)})v^{-1}(h_{(9)}g_{(5)}) \\ &= v(h_{(1)})(h_{(2)} \cdot^u v(g_{(1)}))\omega^u(h_{(3)}, g_{(2)})v^{-1}(h_{(4)}g_{(3)}) \\ &= (\chi^u)^v(h, g) \end{aligned}$$

and thus $\chi^{vu} = (\chi^u)^v$.

Also,

$$\begin{aligned} & h(\cdot^u)^v a \\ &= v(h_{(1)})(h_{(2)} \cdot^u a)v^{-1}(h_{(3)}) \\ &= v(h_{(1)})u(h_{(2)})(h_{(3)} \cdot a)u^{-1}(h_{(4)})v^{-1}(h_{(5)}) \\ &= v(h_{(1)})u(h_{(2)})(h_{(3)} \cdot a)u^{-1}(h_{(4)})v^{-1}(h_{(5)}) \\ &= h \cdot^{vu} a \end{aligned}$$

and so $\cdot^{vu} = (\cdot^u)^v$

This completes the proof. \square

Theorem 5.4. Let H be a Hopf algebra and (A, α, ω) a symmetric twisted partial H -module algebra, and $v \in \text{Hom}(H, A)$ a convolution-invertible linear map, with the above notations ω^v, \cdot^v . Then we have the following assertions:

- (1) As algebras, $A \#_{\alpha, \omega} H \cong A \#_{\alpha, \omega^v} H$;
- (2) ω satisfies Eq.(3.6) if and only if ω^v satisfies Eq.(3.6);

(3) (ω, \cdot) satisfies Eq.(3.7) if and only if (ω^v, \cdot^v) satisfies Eq.(3.7);

(4) If (ω, \cdot) satisfies Eq.(3.7), then (ω, \cdot) satisfies Eq.(3.8) if and only if (ω^v, \cdot^v) satisfies Eq.(3.8);

(5) $A \#_{\alpha, \omega} H$ is a partial crossed product algebra if and only if $A \#_{\alpha, \omega^v} H$ is a partial crossed product algebra, and they are isomorphic.

Proof. (1) Define $\Phi : A \#_{\alpha, \omega} H \mapsto A \#_{\alpha, \omega^v} H$ by $a \# h \mapsto av(h_{(1)}) \# h_{(2)}$, for any $h, g \in H$, we have

$$\begin{aligned}
& \Phi((a \# h)(b \# g)) \\
&= \Phi(a(h_{(1)} \cdot^v b) \omega^v(h_{(2)}, g_{(1)}) \# h_{(3)} g_{(2)}) \\
&= a(h_{(1)} \cdot^v b) \omega^v(h_{(2)}, g_{(1)}) v(h_{(3)} g_{(2)}) \# h_{(4)} g_{(3)} \\
&= av(h_{(1)})(h_{(2)} \cdot b) v^{-1} h_{(3)} v(h_{(4)}) h_{(5)} \cdot \omega(h_{(6)}, g_{(2)}) v^{-1}(h_{(7)} g_{(3)}) v(h_{(8)} g_{(4)}) \# h_{(9)} g_{(5)} \\
&= av(h_{(1)})(h_{(2)} \cdot b)(h_{(3)} \cdot v(g_{(1)})) \omega(h_{(4)}, g_{(2)}) \# h_{(5)} g_{(3)} \\
&= \Phi(a \# h) \Phi(b \# g).
\end{aligned}$$

Clearly Φ is bijective, $\Phi^{-1}(a \# h) = \sum av^{-1}(h_{(1)}) \# h_{(2)}$ $a, b \in A, h \in H$, since

$$\begin{aligned}
& \Phi \Psi(a \# h) \\
&= \Phi(av^{-1}(h_{(1)}) \# h_{(2)}) \\
&= av^{-1}(h_{(1)}) v(h_{(2)}) \# h_{(3)} \\
&= av^{-1}(h_{(1)}) v(h_{(2)}) \# h_{(3)} \\
&= a \# h.
\end{aligned}$$

(2) Straightforward.

(3) If (ω, \cdot) satisfies Eq.(3.7), then

$$\begin{aligned}
& (h_{(1)} \cdot^v (g_{(1)} \cdot^v (a))) \omega^v(h_{(2)}, g_{(2)}) \\
&= v(h_{(1)})(h_{(2)} \cdot (v(g_{(1)})))(g_{(2)} \cdot a) v^{-1}(g_{(3)}) v^{-1}(h_{(3)}) v(h_{(4)}) (h_{(5)} \cdot v(g_{(4)})) \\
& \quad \omega(h_{(6)}, g_{(5)}) v^{-1}(h_{(7)} g_{(6)}) \\
&= v(h_{(1)})(h_{(2)} \cdot v(g_{(1)}))(h_{(3)} \cdot (g_{(2)} \cdot a)) \omega(h_{(4)}, g_{(3)}) v^{-1}(h_{(5)} g_{(4)}) \\
&= v(h_{(1)})(h_{(2)} \cdot v(g_{(1)})) \omega(h_{(3)}, g_{(2)})(h_{(4)} g_{(3)} \cdot a) v^{-1}(h_{(5)} g_{(4)}) \\
&= \omega^v(h_{(1)}, g_{(1)})(h_{(2)} g_{(2)} \cdot^v a).
\end{aligned}$$

Conversely, we get it from Lemma 5.3.

(4) If (ω, \cdot) satisfies Eq.(3.7) and Eq.(3.8), then for $h, g, m \in H$, we have

$$\begin{aligned}
& (h_{(1)} \cdot^v \omega^v(g_{(1)}, m_{(1)})) \omega^v(h_{(2)}, g_{(2)} m_{(2)}) \\
= & v(h_{(1)})(h_{(2)} \cdot [(v(g_{(1)})(g_{(2)} \cdot^v m_{(1)})) \omega(g_{(3)}, m_{(2)}) v^{-1}(g_{(4)} m_{(3)})]) \\
& v^{-1}(h_{(3)}) v(h_{(4)})(h_{(5)} \cdot v(g_{(5)} m_{(4)})) \omega(h_{(6)}, g_{(6)} m_{(5)}) v^{-1}(h_{(7)} g_{(7)} m_{(6)}) \\
= & v(h_{(1)})(h_{(2)} \cdot v(g_{(1)}))(h_{(3)} \cdot (g_{(2)} \cdot v(m_{(1)}))) \\
& (h_{(4)} \cdot \omega(g_{(3)} m_{(2)})) \omega(h_{(5)}, g_{(4)} m_{(3,1)}) v^{-1}(h_{(6)} g_{(5)} m_{(4)}) \\
= & v(h_{(1)})(h_{(2)} \cdot v(g_{(1)}))(h_{(3)} \cdot (g_{(2)} \cdot v(m_{(1)}))) \\
& \omega(h_{(4)}, g_{(3)}) \chi(h_{(5,1)}, g_{(4,1)} m_{(2,1)}) v^{-1}(h_{(6)} g_{(5)} m_{(4)}) \\
= & v(h_{(1,\alpha)})(h_{(2)} \cdot v(g_{(1)})) \omega(h_{(3)} g_{(2)})(h_{(4)} g_{(3)} \cdot v(m_{(1)})) \\
& \omega(h_{(5)} g_{(4)}, m_{(2)}) v^{-1}(h_{(6)} g_{(5)} m_{(4)}) \\
= & \omega^v(h_{(1)}, g_{(1)}) \omega(h_{(2)} g_{(2)}, m).
\end{aligned}$$

Conversely, we get it from Lemma 5.3.

(5) Clearly. □

6 Separable extension for partial crossed products

In this section, we will investigate when $A \subset A \#_{\alpha, \omega} H$ is a separable extension.

Definition 6.1.^[4] Let B be a right H -comodule unital algebra with coaction given by $\rho : B \rightarrow B \otimes H$ and let A be a subalgebra of B . We will say that $A \subset B$ is an H -extension if $A = B^{coH}$. An H -extension $A \subset B$ is partially cleft if there is a pair of k -linear maps $\gamma, \gamma' : H \rightarrow B$ such that

- (i) $\gamma(1_H) = 1_B$,
- (ii) $\rho \circ \gamma = (\gamma \otimes id_H) \Delta$ and $\rho \circ \gamma' = (\gamma' \otimes S) \Delta^{cop}$,
- (iii) $(\gamma * \gamma') \circ M$ is a central element in the convolution algebra $\text{Hom}(H \otimes H, A)$, where $M : H \otimes H \rightarrow H$ is the multiplication in H , and $(\gamma * \gamma')(h)$ commutes with every element of A for each $h \in H$.

Lemma 6.2 Let $c \in C_{A \#_{\alpha, \omega} H}(A)$, the centralizer of A in $A \#_{\alpha, \omega} H$, and assume that H is cocommutative. Then for $h \in H$,

$$\begin{aligned}
\gamma'(h_{(1)}) c(S(h_{(2)}) \cdot 1_A) \gamma(h_{(3)}) &= \gamma(S(h_{(2)})) c \gamma'(S(h_{(1)})) \\
&= S(h) \cdot c.
\end{aligned}$$

Proof. For any $h \in H$, we have

$$\begin{aligned}
& \gamma'(h_{(1)})c(S(h_{(2)}) \cdot 1_A)\gamma(h_{(3)}) \\
&= \gamma'(h_{(1)})c\gamma(h_{(4)})\gamma(S(h_{(3)}))\gamma'(S(h_{(2)})) \\
&= \gamma'(h_{(1)})c\gamma(h_{(2)})\gamma(S(h_{(3)}))\gamma'(S(h_{(4)})) \\
&= \gamma'(h_{(1)})\gamma(h_{(2)})\gamma(S(h_{(3)}))c\gamma'(S(h_{(4)})) \\
&= (S(h_{(1)}) \cdot 1_A)\gamma(S(h_{(2)}))c\gamma'(S(h_{(3)})) \\
&= (S(h_{(1)}) \cdot 1_A)(S(h_{(2)}) \cdot c) \\
&= S(h) \cdot c
\end{aligned}$$

Theorem 6.2. Let H be a finite dimensional cocommutative Hopf algebra and $(A, \cdot, (\omega, \omega'))$ is a symmetric partial twisted H -module algebra, and $t \neq 0$ be a left integral in H . Assume that there exists c in the center of A such that $t \cdot c = 1_A$. Then $A \subset A\#_{\alpha, \omega}H$ is a separable extension.

Proof. In order to show that $A \subset A\#_{\alpha, \omega}H$ is separable, it suffices to find a separability idempotent $e \in (A\#_{\alpha, \omega}H) \otimes_A (A\#_{\alpha, \omega}H)$; this means that

- (1) $((a\#_{\alpha, \omega}h) \otimes_A (1\#_{\alpha, \omega}1))e = e((1\#_{\alpha, \omega}1) \otimes_A (a\#_{\alpha, \omega}h))$ for all $a\#_{\alpha, \omega}h \in A\#_{\alpha, \omega}H$ and
- (2) $m_{A\#_{\alpha, \omega}H}(e) = 1\#_{\alpha, \omega}1$, where $m_{A\#_{\alpha, \omega}H}$ denotes multiplication $(A\#_{\alpha, \omega}H) \otimes (A\#_{\alpha, \omega}H) \rightarrow (A\#_{\alpha, \omega}H)$.

Now since t is a left integral in H , $u = S(t)$ is a right integral in H . We claim that $f = \gamma'(u_{(1)}) \otimes_A \gamma(u_{(2)}) \in (A\#_{\alpha, \omega}H) \otimes_A (A\#_{\alpha, \omega}H)$ satisfies condition (1) above. By [7], one can get the canonical map can:

$$\begin{aligned}
& A\#_{\alpha, \omega}H \otimes_A (A\#_{\alpha, \omega}H) \rightarrow (A\#_{\alpha, \omega}H) \otimes H \\
& (a\#_{\alpha, \omega}h)(b\#_{\alpha, \omega}g) \mapsto (a\#_{\alpha, \omega}h)(b\#_{\alpha, \omega}g_{(1)}) \otimes g_{(2)}
\end{aligned}$$

is a bijection, since by [4] $A \subset A\#_{\alpha, \omega}H$ is a partially cleft extension, we can check $A \subset A\#_{\alpha, \omega}H$ also is partial H -Hopf Galois, we claim that both $((a\#_{\alpha, \omega}h) \otimes_A (1\#_{\alpha, \omega}1))f$ and $f((1\#_{\alpha, \omega}1) \otimes_A (a\#_{\alpha, \omega}h))$ go to $(a\#_{\alpha, \omega}h)(S(u_{(1)}) \cdot 1_A) \otimes u_{(2)}$. As a matter of fact,

$$\begin{aligned}
& can(((a\#_{\alpha, \omega}h) \otimes_A (1\#_{\alpha, \omega}1))f) \\
&= can((a\#_{\alpha, \omega}h)\gamma'(u_{(1)}) \otimes_A \gamma(u_{(2)})) \\
&= (a\#_{\alpha, \omega}h)\gamma'(u_{(1)})\gamma(u_{(2)}) \otimes u_{(3)} \\
&= (a\#_{\alpha, \omega}h)(S(u_{(1)}) \cdot 1_A) \otimes u_{(2)},
\end{aligned}$$

and

$$\begin{aligned}
& \text{can}((f(a\#_{\alpha,\omega}h) \otimes_A (1\#_{\alpha,\omega}1))) \\
&= \text{can}(\gamma'(u_{(1)}) \otimes_A \gamma(u_{(2)})(a\#_{\alpha,\omega}h)) \\
&= \text{can}(\gamma'(u_{(1)}) \otimes_A (u_{(2)} \cdot a)\omega(u_{(3)}, h_{(1)})\gamma(u_{(4)}h_{(2)})) \\
&= \gamma'(u_{(1)})(u_{(2)} \cdot a)\omega(u_{(3)}, h_{(1)})\gamma(u_{(4)}h_{(2)}) \otimes u_{(5)}h_{(3)} \\
&= \gamma'(u_{(1)})\gamma(u_{(2)})a\gamma'(u_{(3)})\omega(u_{(4)}, h_{(1)})\gamma(u_{(5)}h_{(2)}) \otimes u_{(6)}h_{(3)} \\
&= (S(u_{(1)}) \cdot 1_A)a\gamma'(u_{(2)})\omega(u_{(3)}, h_{(1)})\gamma(u_{(4)}h_{(2)}) \otimes u_{(5)}h_{(3)} \\
&= (S(u_{(1)}) \cdot 1_A)a\gamma(h_{(1)})\gamma'(u_{(2)}h_{(2)})\gamma(u_{(3)}h_{(3)}) \otimes u_{(4)}h_{(4)} \\
&= (S(u_{(1)}) \cdot 1_A)a\gamma(h_{(1)})\gamma'(u_{(2)})\gamma(u_{(3)}) \otimes u_{(4)} \\
&= (a\#_{\alpha,\omega}h)(a\#_{\alpha,\omega}h)(S(u_{(1)}) \cdot 1_A) \otimes u_{(2)}.
\end{aligned}$$

Now since c is in the center of A , the map

$$\begin{aligned}
& (A\#_{\alpha,\omega}H) \otimes_A (A\#_{\alpha,\omega}H) \rightarrow (A\#_{\alpha,\omega}H) \otimes_A (A\#_{\alpha,\omega}H) \\
& (a\#_{\alpha,\omega}h) \otimes_A (b\#_{\alpha,\omega}g) \mapsto (a\#_{\alpha,\omega}h)c \otimes_A (b\#_{\alpha,\omega}g)
\end{aligned}$$

is well-defined. Thus $e = \gamma'(u_{(1)})c(S(u_{(2)}) \cdot 1_A) \otimes_A \gamma(u_{(3)})$ also satisfies (1).

We also claim that $m(e) = \gamma'(u_{(1)})c(S(u_{(2)}) \cdot 1_A)\gamma(u_{(3)}) = 1_A$. For, by Lemma 6.1,

$$\gamma'(u_{(1)})c(S(u_{(2)}) \cdot 1_A)\gamma(u_{(3)}) = S(u) \cdot c = S^2(t) \cdot c = t \cdot c = 1_A.$$

It follows that $A \subset A\#_{\alpha,\omega}H$ is a separable extension.

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